Continuous Unitary Transformations Approach to One Dimensional Quantum Ising Model With Nearest Neighbor Ferromagnetic Interaction

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- *One Dimensional Quantum Ising Model*
- *Continuous Unitary Transformations (CUT) Method*
- *Applying CUT Method to One Dimensional Quantum Ising Model with Nearest Neighbor (NN) ferromagnetic Interaction.*

1D Quantum Ising Model With (NN) Ferromagnetic Interaction

- *Hamiltonian* :

$$H = -J \sum_{n} g\sigma_n^x - J \sum_{<n,m>} \sigma_n^z \sigma_m^z$$

where:

J ≡ *Exchange Constant (J>0)*

g ≡ *Dimensionless Coupling Constant*

$\sigma^{x,z}$ ≡ *Pauli Matrices*

$\langle m,n \rangle$ ≡ *denotes that the sum is over pairs of nearest neighbor sites m,n*

Studying Ising Hamiltonian (NN) in Two Limits : g››1 & g‹‹1

- Hamiltonian:

$$H = -J\sum_{n} g\sigma_n^x - J\sum_{<n,m>} \sigma_n^z \sigma_m^z$$

- In g››1 :

Eigen State : $|0\rangle = \prod_n |\rightarrow\rangle_n$ where $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$

- In g‹‹1 :

Eigen State : $|0\rangle = \prod_n |\uparrow\rangle_n$ or $|0\rangle = \prod_n |\downarrow\rangle_n$

There should be a phase transition between Paramagnetic and Ferromagnetic phase at $g = g_c$

Exact Solution of 1D Quantum Ising With(NN) Ferromagnetic Interaction

- ***Step 1 : Jordan-Wigner Transformation***

- A powerful mapping between models with spin-1/2 degree of freedom and auxiliary spinless fermions.

- On a single site : Spin up state ↔ no fermion

Spin down state ↔ occupied with a fermion

$$\begin{cases} \sigma_i^- = \prod_{i<j}(1-2C_j^\dagger C_j)C_i^\dagger \\ \sigma_i^+ = \prod_{i<j}(1-2C_j^\dagger C_j)C_i \end{cases} \quad \longleftrightarrow \quad \begin{cases} \sigma_i^z = -\prod_{j<i}(1-2C_j^\dagger C_j)(C_i^\dagger + C_i) \\ \sigma_i^x = 1-2C_i^\dagger C_i \end{cases}$$

- ***Step2 : Furrier Transformation :***

$$C_k = \frac{1}{\sqrt{M}} \sum_j C_j e^{ikr_j}$$

- ***Step3 : Bogoliubov Transformation :***

- *we use Bogoliubov transformation to map a new set of fermionic operators (γ_k) whose numbers is conserved.*

$$\boldsymbol{\gamma_k = u_k C_k - i v_k C_k^\dagger}$$

Where :

1. *u and v are real numbers*
2. $u^2_k + v^2_k = 1$
3. $u_{-k} = u_k$ & $v_{-k} = v_k$

Exact Spectrum of 1D Quantum Ising With (NN) Ferromagnetic Interaction

After these steps we can derive for 1D quantum Ising model Hamiltonian :

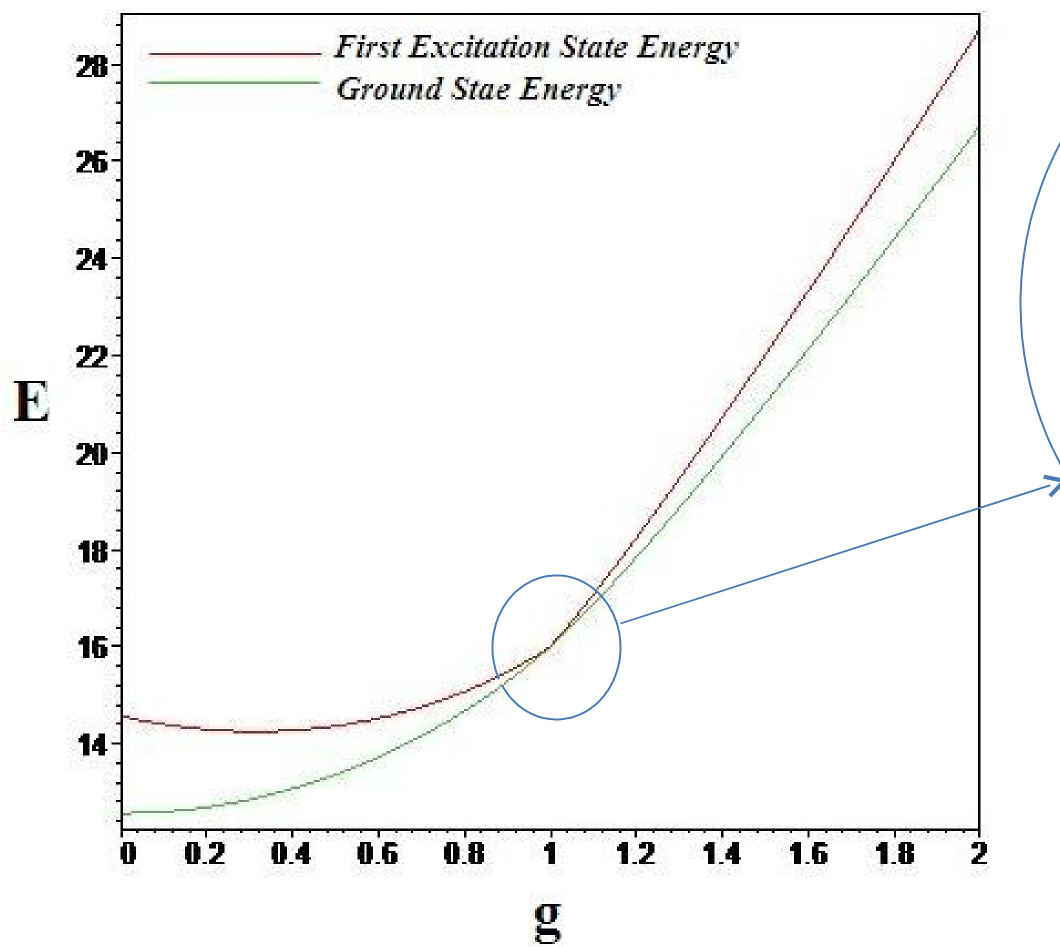
$$H_I = \sum_k \varepsilon_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2}\right)$$ where $$\varepsilon_k = 2J\sqrt{1 + g^2 - 2g\cos(ka)}$$

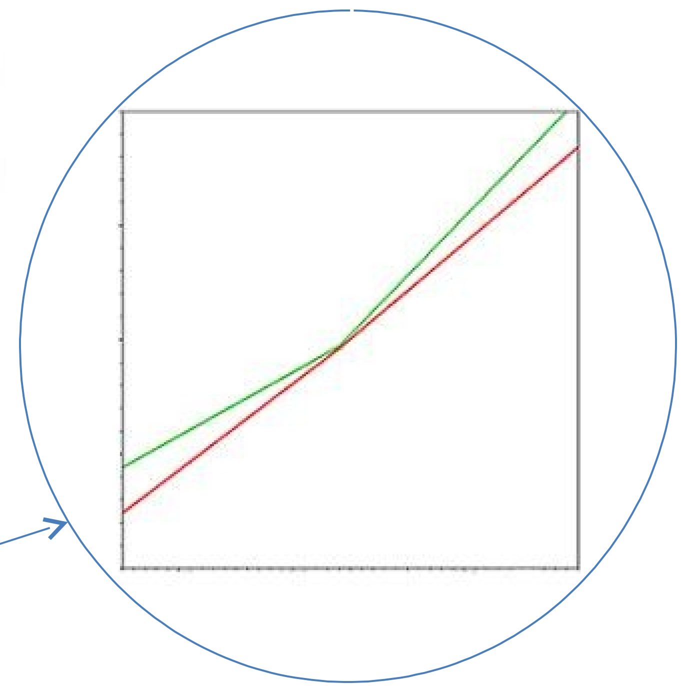
Ground state energy and first excitation state energy are :

$$E_0 = -\frac{1}{2}\sum_k \varepsilon_k \qquad E_1 = E_0 + 2J|1 - g|$$

Second order Quantum phase transition

- A Ferromagnetic to paramagnetic second order phase transition in g = 1.





$$E \sim |g - g_c|^{\nu} \Rightarrow \nu = 1$$

Theoretical Methods for Solving Quantum Many-Body Problems

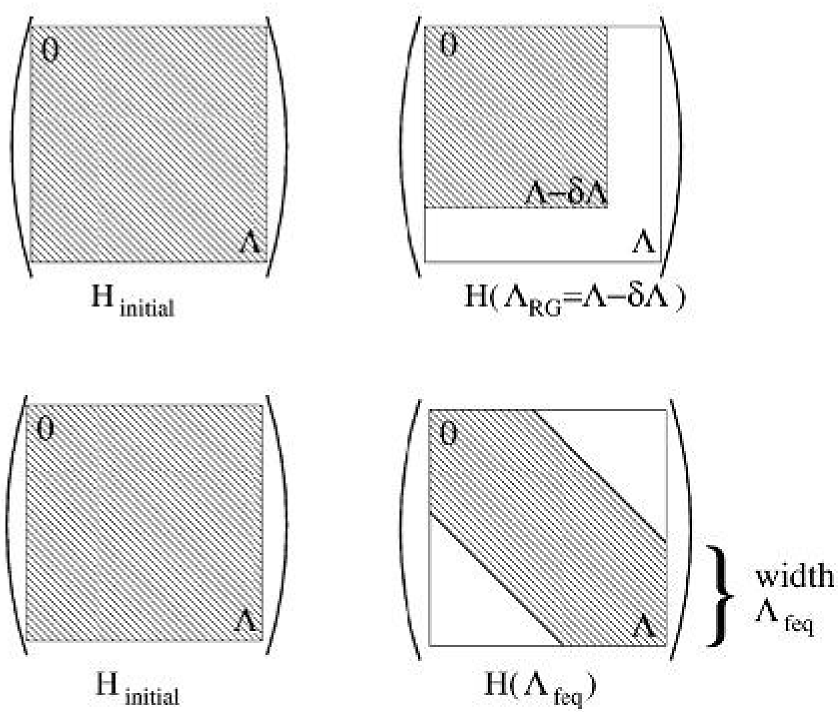
- Exact analytical solution
- Perturbative analytical expansions
- Renormalization group analytical solution
- Numerical solution using computers

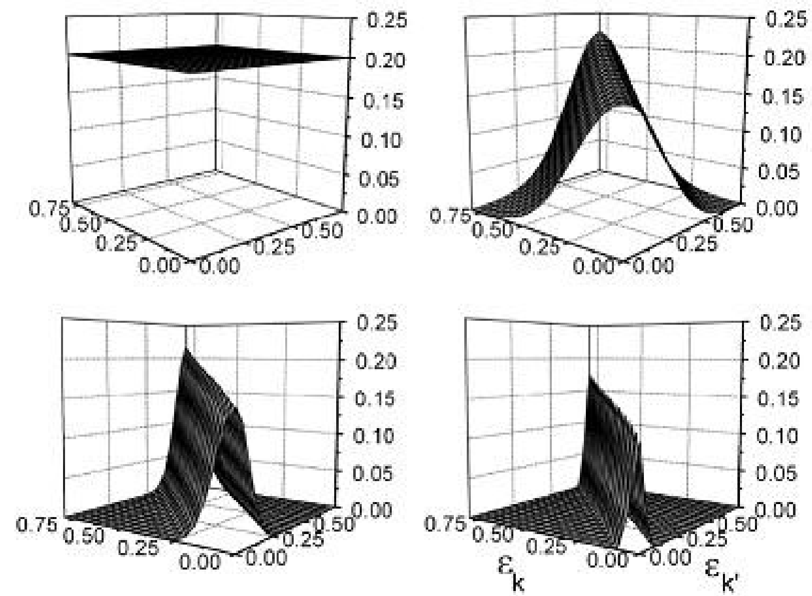
CUT as a new method for solving Quantum Many-Body problems

- ***Motivation :*** Many condensed matter systems that involve strong electron correlations like in high temperature superconductors.
- ***History of this method*** : in 1994 Glazek and Wilson in Particle Physics and independently Wegner in Condensed matter physics introduced this method.
- ***Basic ideas* : 1.** Instead of diagonalizing of Hamiltonian by a single unitary transformation, perform *a continuous sequence of infinitesimal unitary transformation and thus induces flow on the system parameters*.

2. *Retaining the full Hilbert space.*

CUT with pictures :





CUT Method

- Consider infinitesimal unitary transformation :

$$H \longrightarrow H(B)\ ,\ B \in R^{+}$$

$$H(B) = U(B)\, H(0) U^{\dagger}(B)$$

$$\frac{dH(B)}{dB} = \frac{dU(B)}{dB} H(0)U^{\dagger}(B) + U(B)H(0)\frac{dU^{\dagger}(B)}{dB}$$

Where $U = e^{\eta}$

Introduce an infinitesimal *anti-hermitian generator* $\eta = -\eta^{\dagger}$. *η may also be a function of B, therefore :*

$$\frac{dU(B)}{dB} = \eta(B)U(B)$$

We derive this relation so-called ***flow equation*** :

$$\frac{dH(B)}{dB} = [\eta(B), H(B)]$$

- ***Choice of the generator*** is at the heart of the flow equation method.

- ***Wegner generator*:**

$$H = H_0 + H_{int} \Rightarrow \eta(B) = [H_0, H_{int}]$$

- With this choice of generator we can derive flow equations :

$$\eta_{ij} = (\varepsilon_i - \varepsilon_j) h_{ij}$$

$$\frac{dh_{ii}}{dB} = 2 \sum_k (\varepsilon_i - \varepsilon_k) |h_{ik}|^2$$

$$\frac{dh_{ij}}{dB} = -(\varepsilon_i - \varepsilon_j)^2 h_{ij} + \sum_k (\varepsilon_i + \varepsilon_j - 2\varepsilon_k)\, h_{ik} h_{kj}$$

Advantage of CUT Method

1. ***Observables*** can be calculated:

$$\langle O\rangle_{gs} = \langle\psi_{gs}(B=0)|O(B=0)|\psi_{gs}(B=0)\rangle$$
$$= \langle\psi_{gs}(B=\infty)|O(B=\infty)|\psi_{gs}(B=\infty)\rangle$$

2. ***Correlation functions, green functions*** can also be calculated.*

3. ***Real time evolution of quantum many body systems***.**

*Flow equation approach to many-particle systems (Springer, 2006)

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Applying CUT Method to 1D Quantum Ising Model (NN)

- *For getting exact spectrum, we simplified the 1D quantum Ising Hamiltonian with Jordan-Wigner and Furrier transformations :*

$$H_{ising}(B) = \sum_k \{2J_k(B)[g_k(B) - cos(ka)]C_k^\dagger C_k + iJ_k sin(ka)(C_{-k}C_k + C_{-k}^\dagger C_k^\dagger) - J_k g_k\}$$

$$\eta_k = 2i\sum_k \{J_k(g_k - cos(ka))(J_k + J_{-k})(C_{-k}^\dagger C_k^\dagger - C_{-k}C_k)\}$$

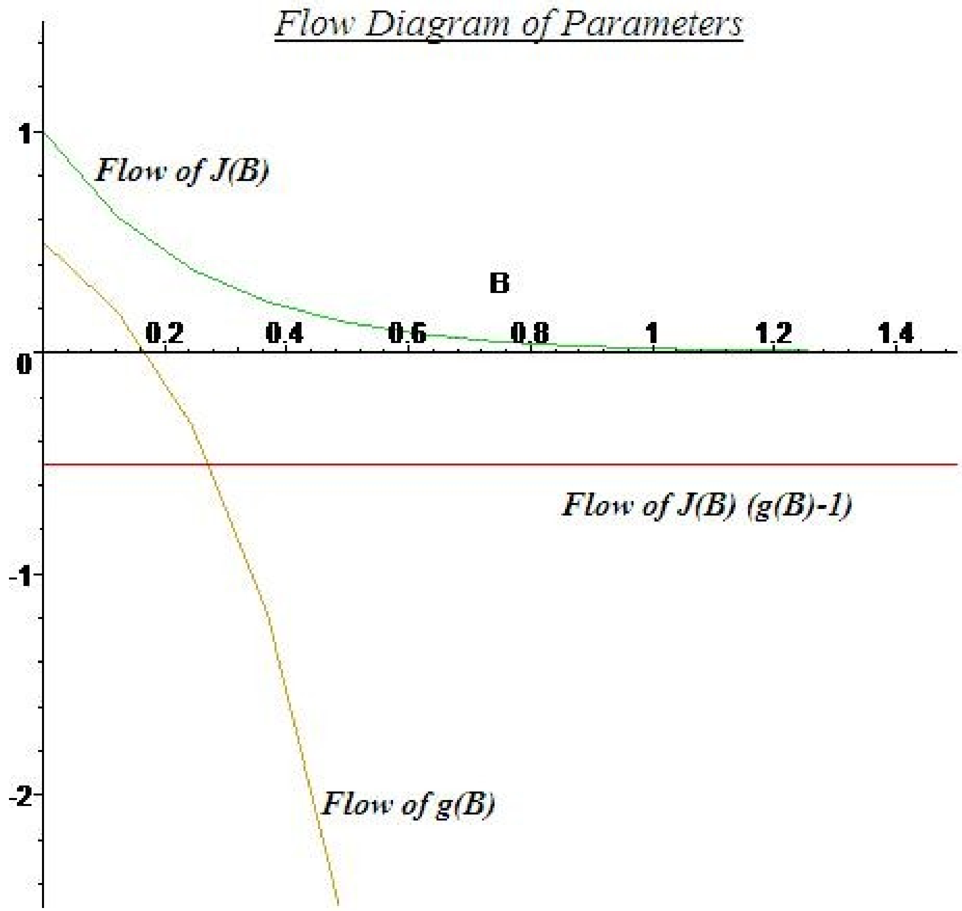
- *Flow equations :* $\begin{cases} \frac{d}{dB}\{J_k(g_k - \cos(ka))\} = 16\, J_k^3(g_k - \cos(ka))sin^2(ka) \\ \frac{d\, J_k}{dB} = -16J_k^3(g_k - \cos(ka))^2 \end{cases}$

- *With solving these equations, in limit of B→∞ we can derive exact spectrum:*

$$\lim_{B\to\infty}\{2J_k(B)(g_k(B) - cos(ka))\} = 2J_k(B=0)\sqrt{g_k^2(B=0) - 2g_k(B=0)\,cos(ka) + 1}$$

$$\lim_{B\to\infty} J_k(B) \to 0$$

- ***The flow of parameters for k =0 & J(B =0)=1 & g(B =0)=0.5 :***



- *To derive expectation value of σ^x and σ^z we have to calculate the flow of these operators :*

$$C_k(B)=\frac{1}{2}\left(C_k(0)+C^{\dagger}_{-k}(0)\right)exp(i\Gamma_k)+\frac{1}{2}\left(C_k(0)-C^{\dagger}_{-k}(0)\right)exp(-i\Gamma_k)$$
$$C^{\dagger}_k(B)=\frac{1}{2}\left(C_{-k}(0)+C^{\dagger}_k(0)\right)exp(-i\Gamma_k)-\frac{1}{2}\left(C_{-k}(0)-C^{\dagger}_k(0)\right)exp(i\Gamma_k)$$

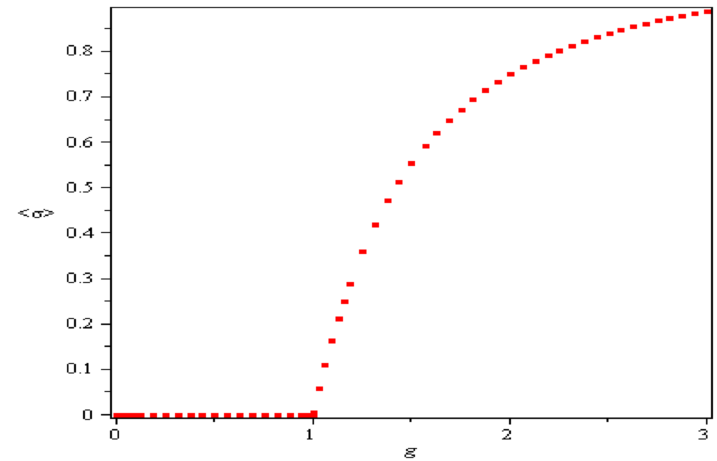
Where in these relations :

$$\Gamma_k=\int_0^B 8\,J_k^2(B)(g_k(B)-\cos(ka))\sin(ka)dB$$

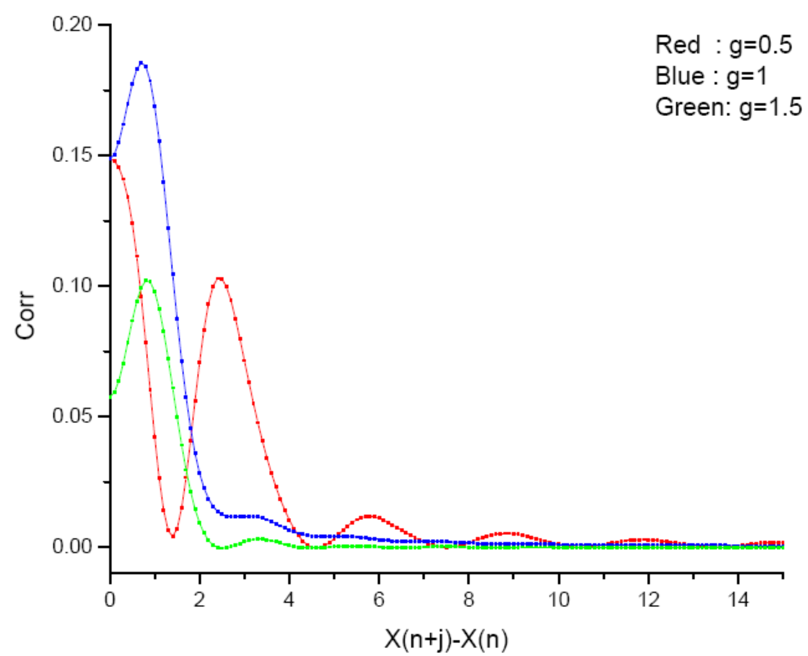
According to above equations :

$$\langle\sigma_n^z\rangle=\langle-\prod_{m<n}\left(1-2C^{\dagger}_m(B=\infty)C_m(B=\infty)\right)(C^{\dagger}_n(B=\infty)+C_n(B=\infty))\rangle=0$$

$$\langle \sigma_n^x \rangle = \langle (1 - 2C_n^\dagger(B=\infty)C_n(B=\infty)) \rangle = \begin{cases} 0 & g \le 1 \\ 1 - \dfrac{1}{g^2} & g > 1 \end{cases}$$



$$\langle \sigma_n^x(B=0)\sigma_{n+j}^x(B=0) \rangle_{B=0} = \langle \sigma_n^x(B=\infty)\sigma_{n+j}^x(B=\infty) \rangle_{B=\infty}$$



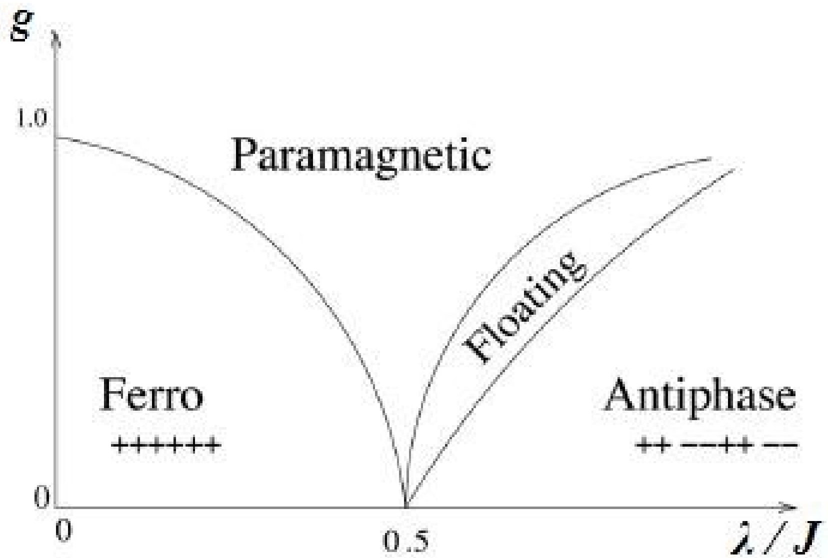
Thanks

1D Quantum Ising Model with Nearest Neighbor Ferromagnetic and Next Nearest Antiferromagnetic Interaction

- *Hamiltonian :*

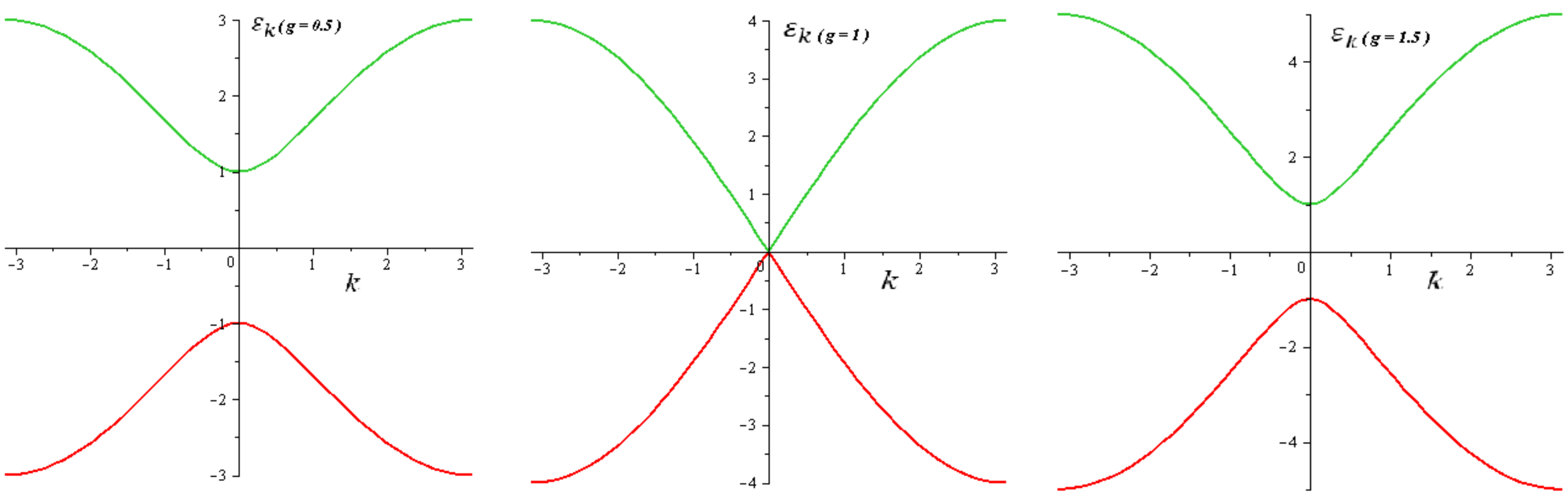
$$H = -J\sum_{n} g\sigma_n^x - J\sum_{\langle n,m\rangle}\sigma_n^z\sigma_m^z + \lambda\sum_{\langle\langle n,m\rangle\rangle}\sigma_n^z\sigma_m^z$$

- *Phase diagram:*



Questions

- *Commutation of Furrier transformations and CUT*
- *Difference between Heisenberg equation and flow equation*
- *Dependency of couplings on sites (in real space) or momentum (in Furrier space)*



$\varepsilon_k (g=0.5)$
k
$\varepsilon_k (g=1)$
k
$\varepsilon_k (g=1.5)$
k

3. Real Time Evolution of Quantum Many Body Systems :